1 a) Define $p(x) := 1$ and $q(x) := x$.

Then $\tilde{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\tilde{q} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Note that $\tilde{p}^T \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 0$ and $\tilde{q}^T \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 0$.

Also, for $r(x) = -2 + 3x^2$, $\tilde{r} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

So r is orthogonal to both p and q.

b) Let $p \in \mathcal{E}$ and $q \in \mathcal{O}$. Since p is even, $p(-1) = p(1)$ so

$$\tilde{p} = \begin{pmatrix} p(0) \\ p(1) \\ p(1) \end{pmatrix}.$$

Moreover, because q is odd, $q(0) = -q(0)$ so $q(0) = 0$, and $q(-1) = -q(1)$.

Hence,

$$\tilde{q} = \begin{pmatrix} 0 \\ q(1) \\ -q(1) \end{pmatrix}.$$

Therefore, $\tilde{p}^T \tilde{q} = p(1)q(1) - p(1)q(1) = 0$.

We conclude that $\mathcal{E}$ and $\mathcal{O}$ are orthogonal.

c) By b), $\mathcal{O} \subseteq \mathcal{E}^\perp$. It remains to be shown that $\mathcal{E}^\perp \subseteq \mathcal{O}$, which can be done in different ways.

For example, let $p \in \mathcal{E}^\perp$. Then p is orthogonal to all polynomials in $\mathcal{E}$, in particular to $x^2 \in \mathcal{E}$

and $1 \in \mathcal{E}$. Write $p(x) = p_0 + p_1 x + p_2 x^2$ with $p_0, p_1, p_2 \in \mathbb{R}$ and note that the latter implies that

$$\begin{pmatrix} p_0 \\ p_0+p_1+p_2 \\ p_0-p_1+p_2 \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \text{and}$$

$$\begin{pmatrix} p_0 \\ p_0+p_1+p_2 \\ p_0-p_1+p_2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0.$$

Hence, $2(p_0+p_2)=0$ and $3p_0+2p_2=0$.

$\Rightarrow p_0 = 0$ and thus also $p_2 = 0$.

So p is of the form $p(x) = p_1 x$, which is indeed odd. $\Rightarrow \mathcal{E}^\perp \subseteq \mathcal{O}$ and thus $\mathcal{O} = \mathcal{E}^\perp$.

4 a)

$$A^TA = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}.$$

Now compute the eigenvalues of A^TA:

$$\det \begin{pmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = (8-\lambda)(5-\lambda) - 4 = 0 \Rightarrow$$

$$\lambda^2 - 13\lambda + 36 = 0 \Rightarrow (\lambda - 9)(\lambda - 4) = 0$$

$\Rightarrow \lambda_1 = 9$ and $\lambda_2 = 4$.

We conclude that the singular values of A are:

$\sigma_1 = 3$ and $\sigma_2 = 2$.

Hence, Σ is of the form

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Next, we compute a V:

$$\begin{pmatrix} 8-9 & 2 \\ 2 & 5-9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies 2y - x = 0 \text{ and}$$

$$2x - 4y = 0 \implies x = 2y.$$

Therefore, choose

$$V_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

so that $\|V_1\| = 1$.

Moreover,

$$\begin{pmatrix} 8-4 & 2 \\ 2 & 5-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies 4x + 2y = 0$$

and $2x + y = 0 \implies y = -2x$.

So pick

$$V_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Hence, $V = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$

Now, $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \frac{1}{3\sqrt{5}} \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$

Also, $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$= \frac{1}{2\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix}.$

Finally, we compute u_3, which has to be of norm 1 and orthogonal to both u_1 and u_2.

Write $u_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $2y - 4z = 0$ and $5x + 4y + 2z = 0$.

So u_3 is of the form

$u_3 = \begin{pmatrix} -2y \\ 2y \\ y \end{pmatrix}$. We take $u_3 = \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ so that

$\|u_3\| = 1$. Therefore,

$$\mathcal{U} = \begin{pmatrix} \frac{5}{3\sqrt{5}} & 0 & -\frac{2}{3} \\ \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{-2}{\sqrt{5}} & \frac{1}{3} \end{pmatrix}$$

A singular value decomposition of A is then:

$A = U \Sigma V^T$, i.e.,

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3\sqrt{5}} & 0 & -\frac{2}{3} \\ \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{-2}{\sqrt{5}} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

b) $d(A, M_1) = \sigma_2 = 2$ and a best rank-1 approximation is:

$$X = \begin{pmatrix} \frac{5}{3\sqrt{5}} & 0 & -\frac{2}{3} \\ \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{-2}{\sqrt{5}} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & 0 \\ \frac{4}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & 0 \\ \frac{2}{5} & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ \frac{8}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{2}{5} \end{pmatrix}$$